ON A SURFACE SINGULAR BRAID MONOID

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Abstract. We introduce a monoid corresponding to knotted surfaces in four space, from its hyperbolic splitting represented by marked diagram in braid like form. It has four types of generators: two standard braid generators and two of singular type. Then we state relations on words that follows from topological Yoshikawa moves. As a direct application we will show equivalence of some known theorems about twist-spun knots. We wish then to investigate an index associated to the closure of surface singular braid monoid. Using our relations we will prove that there are exactly six types of knotted surfaces with index less or equal to two, and there are infinitely many types of knotted surfaces with index equal to three. Towards the end we will construct a family of classical diagrams such that to unlink them requires at least four Reidemeister III moves.

1. Introduction

We introduce a monoid SSB_m corresponding to knotted surfaces in four space, from its hyperbolic splitting represented by marked diagram in braid like form on m strands. It has four types of generators: two standard c_i and c_i^{-1} braid generators and two noninvertible a_i and b_i of singular type. Then we state 12 relations on words that follows from topological Yoshikawa moves from his paper [12] and other interesting relations.

As a rather direct application we will give algebraic formula for twist-spun knots and show equivalence of some known theorems of Zeeman and Litherland. We wish then to investigate an index associated to the closure of surface singular braid monoid. Using our relations we will prove that there are exactly six types of knotted surfaces with index less or equal to two, and there are infinitely many types of knotted surfaces with index equal to three (representing two-twist-spun of torus (2,k) knots).

In the paper [4] there are given three pairs of diagrams of classical links such that deforming one of them to the other, requires minimum 2 (or 3 in other cases) Reidemeister III moves. We will give another family with similar properties but with completely different proof method.

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2. Basic definitions

We will work in the smooth category, i.e we will be assuming that all manifolds and functions between them are smooth. An image of an embedding of a closed surface to $\mathbb{R}^4$ is called the *knotted surface*. We will use a word: *classical*, thinking about theory of embeddings of circles $S^1 \sqcup \ldots \sqcup S^1 \hookrightarrow \mathbb{R}^3$ modulo ambient isotopy in $\mathbb{R}^3$. Two knotted surfaces are *equivalent* or have the same *type*, if there exists an orientation preserving auto-homeomorphism of $\mathbb{R}^4$, taking one of those surfaces to the other. Without lost of generality we may assume that the image of projection $\pi(x_1, x_2, x_3, t) = (x_1, x_2, x_3)$ is in an *general position*, i.e. the double point set of a surface consists of points whose neighborhood is locally homeomorphic to:

(i) two transversely intersecting sheets,
(ii) three transversely intersecting sheets,
(iii) the Whitney's umbrella.

Points corresponding to cases (i), (ii), (iii) are called: *double point, triple point* and a *branch point* respectively, of the projection.

Let $\mathbb{R}^3_t$ denotes $\mathbb{R}^3 \times \{t\}$ for $t \in \mathbb{R}$. For a surface $K \subset \mathbb{R}^4$, a family $\{K \cap \mathbb{R}^3_t\}_{t\in\mathbb{R}}$ is called *motion picture* or simply *movie* for K. Moreover $K \cap \mathbb{R}^3_t$ is a *still* of that movie. Every knotted surface gives us a movie, and from a specific finite number of stills we can recreate completely the type of corresponding knotted surface.

Proposition 2.1 ([5, p. 12])**.** *In the generic projection of a movie of a knotted surface, Reidemeister III move on stills corresponds to triple point of the knotted surface in projection to* $\mathbb{R}^3$.

More basic terminology and properties may be found in book [5].

2.1. Twist-spun knots.

One of the main family (besides ribbon surface) as an object of study, in knotted surface papers, are *twist-spun knots* that can be defined as follows.

Definition 2.2 ([13])**.** We think of $\mathbb{R}^4$ as an open book decomposition, that is a spun (in the fixed direction) of $\mathbb{R}^3_+$ along $\mathbb{R}^2$. For a classical knot K we take its *tangle* T i.e. properly embedded arc in $\mathbb{R}^3_+$, with distinct end points $a, b \in \partial\mathbb{R}^3_+$ such that $T \cup [a, b] \subset \mathbb{R}^3$ is a knot of the same type as K. Then the geometrical trace of spinning of T along $\mathbb{R}^2$ with additional twisting it in the meanwhile (in the fixed direction) m times in a surrounding ball B^3 we call *m-twist spun* surface knot and denote it by $\tau^m(K)$.

Theorem 2.3 (Zeeman [13])**.** *For every classical knot* K *and* $m \in \mathbb{Z}$*, we have property that* $\tau^{-m}(K) = \tau^m(K)$.

Theorem 2.4 (Litherland [7] (version for non-oriented surfaces)). *For every classical knot K and $m \in \mathbb{Z}$, we have property that $(\tau^m(K))^* = \tau^m(K)$, where X^* denotes taking the mirror image of X.*

2.2. Hyperbolic splitting and marked diagrams.

Theorem 2.5 (Lomonaco [8]). *For every knotted surface $F \subset \mathbb{R}^4$ there exists equivalent knotted surface F' having finitely many Morse's critical points, such that all of its saddle points lie in the hyperspace $\mathbb{R}^3_0$, maxima lie in the hyperspace $\mathbb{R}^3_1$ and minima lie in the hyperspace $\mathbb{R}^3_{-1}$. We call this situation a* hyperbolic splitting *of the surface F.*

The zero section $\mathbb{R}^3_0 \cap F'$ of the hyperbolic splitting gives us a 4-valent graph (with possible loops without vertices). We assign to each vertex a *marker* that informs us about one of the two possible types of saddle points (see Figure 1). Making now a projection in general position of this graph to $\mathbb{R}^2$, and imposing a crossing types like in the classical knot case, we receive a *marked diagram* (terminology also apply to graphs of that kind which do not come from slicing closed surface).

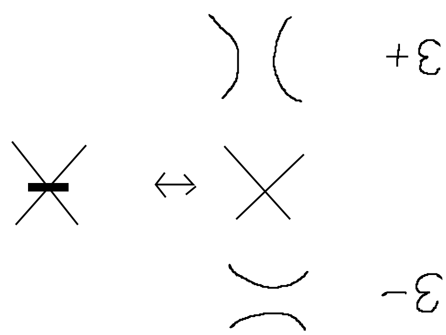

FIGURE 1. resolution of a marker

Theorem 2.6 (cf. [12]). *Every knotted surface can be presented as a marked diagram, and from any of its marked diagram, we can receive all information about knotted surface type from which it was created.*

We have the following characterization of marked diagrams corresponding to knotted surfaces.

Theorem 2.7 (cf. [12]).
Marked diagram D is a diagram of some knotted surface in $\mathbb{R}^4$ if and only if $L_+(D)$ and $L_-(D)$ are planar diagrams of trivial (i.e. unknotted) classical links in $\mathbb{R}^3$, where $L_+(D)$ is obtained from D by removing each vertex by smoothing surrounding arcs in the convention shown in Figure 1 in case if $+\epsilon$; smoothing vertices like in the $-\epsilon$ case gives us diagram $L_-(D)$.

Theorem 2.8 (Swenton [11])**.** *(question asked by Yoshikawa in* [12]*)*

Two knotted surfaces are equivalent if and only if, their marked diagrams may be transformed one to another by isotopy in $\mathbb{R}^2$ *and finite sequence of elementary local moves* $Y_1, \ldots, Y_8$ *taken from the list in the Figure 2 (their mirror moves and moves having all its markers (in that fragment of a diagram) switched to its second type).*

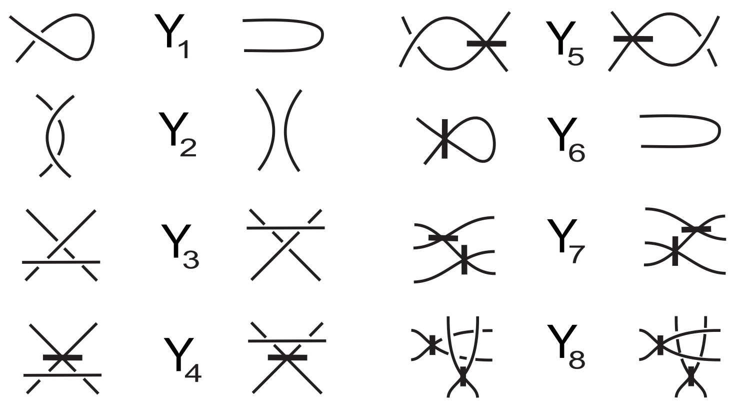

FIGURE 2. Yoshikawa moves

3. MONOID OF KNOTTED SURFACES

Definition 3.1. We say that marked diagram is in *braid form* if by forgetting about marker types, it is the geometric closure of some singular braid (notion that was developed independently in [2] and [3]) of degree m for some $m \in \mathbb{Z}_+$ (i.e. having m numbered strands).

Proposition 3.2.
For every knotted surface there exists its marked diagram in braid form.

Proof. Forgetting for a moment about markers and leaving singular points at its place, we apply the Alexander's like theorem for singular braids from [3]. Moreover we deduce that it is only required to use moves of type $Y_1, Y_2, \ldots, Y_5$ to do a braid form. Putting now markers back in appropriate manner to vertices, we receive a braid form of marked diagram.

□

We now introduce monoid SSB_m corresponding to marked diagrams in braid form on m strands. Elements of that monoid, called *singular surface braids* are generated by four types of elements a_i, b_i, c_i, c_i^{-1} for $i = 1, \ldots, m-1$, where the correspondence of types of crossings and types of markers between i-th and $i+1$-th strand (in the horizontal position) is presented in Figure 3 (remaining strands in braid are strait lines without crossings).

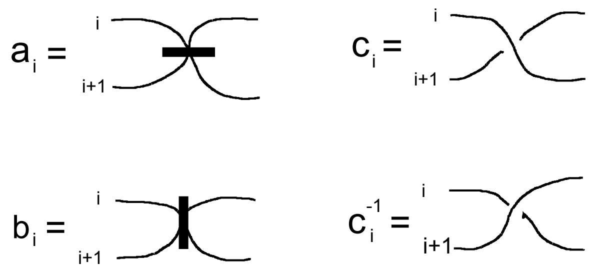

FIGURE 3. correspondence of monoid generators

Our closure of marked diagram in braid form we will indicate by adding brackets [] and sometimes adding lower index to it, saying how many strands we are joining

Example 3.3. We have two types of trivially knotted projective planes $[c_1 a_1]$ and $[c_1^{-1} a_1]$. Standard torus $\mathbb{T}^2$ can be presented as $[b_1 a_1]$.

Let the symbol Δ_s means (known from braid theory) positive half-twist in $\mathbb{R}^3$ of s strands involved in the equation we are concerning (it is clear from words indices). It contains only product of generators of type c_i.

Definition 3.4. Let $m \in \mathbb{Z}_+$ and $i, k, n \in \{1, \ldots, m-1\}$ such that $|k - i| = 1$, moreover let $x_i, y_i \in \{a_i, b_i, c_i, c_i^{-1}\}$. In monoid SSB_m we introduce following relations.

A1) $c_i c_i^{-1} = 1$
A2) $x_i y_n = y_n x_i$ for $n \neq k$
A3) $c_i x_k c_i^{-1} = c_k^{-1} x_i c_k$
A4) $x_i c_k c_i = c_k c_i x_k$
A5) $x_i c_k^{-1} c_i^{-1} = c_k^{-1} c_i^{-1} x_k$
A6) $a_i b_k = b_k a_i$
A7) $a_i b_{i-2} (c_{i-1} c_{i-2} c_i c_{i-1})^2 = a_i b_{i-2}$ for $i > 2$
A8) $b_i a_{i-2} (c_{i-1} c_{i-2} c_i c_{i-1})^2 = b_i a_{i-2}$ for $i > 2$
A9) $a_i^2 = a_i$
A10) $b_i^2 = b_i$
A11) $a_i b_i c_i^2 = a_i b_i$
A12) $a_i b_k \Delta_3 = a_i b_k \Delta_3^{-1}$

Let us denote by CSB_m a subset of SSB_m containing only those elements x, that $L_+([x])$ and $L_-([x])$ are diagrams of trivial classical links.

Definition 3.5. We define moreover following Markov type relations (where $n \in \mathbb{Z}_+$, $x_i \in \{a_i, b_i, c_i, c_i^{-1}\}$).

C1) $[x_i S_n]_n = [S_n x_i]_n$ for $i < n$ and $x_i S_n \in CSB_n$
C2) $[S_n]_n = [S_n x_n]_{n+1}$ for $S_n \in CSB_n$

Theorem 3.6. *Making change in algebraic formulation of knotted surface by using one of relations A1)-A12) or C1)-C2) on words, we receive a formula of equivalent knotted surface.*

Proof. Algebraic relations A1)-A11), C1)-C2) were deduced from topological local moves preserving type of knotted surfaces from paper [12]. The proof of relation A12) for $k = i+1$ (for $k = i-1$ it may be done by analogy, changing marker types and using relation A6)):

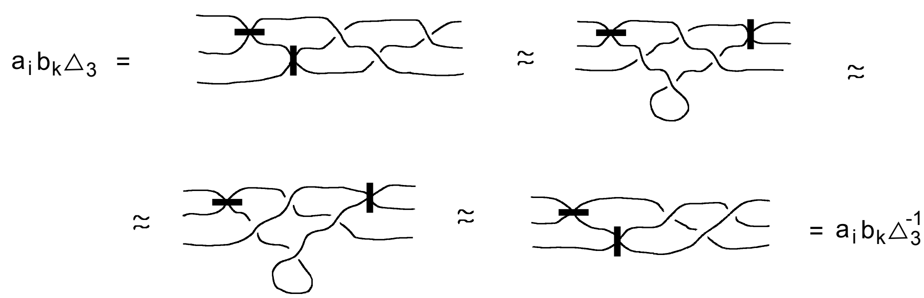

FIGURE 4. A12) move

□

Remark 3.7. It is still open problem, whether any pair of marked diagrams in braid form of equivalent knotted surface, can be transformed one another by using only relations A1)-A12) and C1)-C2).

Lemma 3.8. *Under the assumptions of Definition 3.4 the following relation holds:*

A13) $x_i\Delta_n = \Delta_n x_{n-i}$ *for* $1 \leq i < n \leq m$.

Proof. It follows directly from geometric observation after making positive half-twist in $\mathbb{R}^3$ of n strands with element x_i.

□

Proposition 3.9 (J. H. Przytycki)**.** *Under the assumptions of Definition 3.4 the following relation holds:*

A14) $a_i c_k^{-1} b_i c_k \Delta_3^2 = a_i c_k^{-1} b_i c_k$

Proof. $a_i c_k^{-1} b_i c_k \Delta_3^2 \overset{A13)}{=} a_i c_k^{-1} \Delta_3^2 b_i c_k = a_i c_k^{-1} (c_i c_k c_i)^2 b_i c_k \overset{A4)}{=} a_i c_k^{-1} c_k c_i c_k c_i c_k c_i b_i c_k \overset{A1), A2)}{=} a_i c_i c_k c_i c_k b_i c_i c_k \overset{A2), A4)}{=} c_i a_i \Delta_3 b_i c_i c_k \overset{A13)}{=} c_i a_i b_k \Delta_3 c_i c_k \overset{A12)}{=} c_i a_i b_k (\Delta_3)^{-1} c_i c_k \overset{A2)}{=} a_i c_i c_i^{-1} c_k^{-1} c_i^{-1} c_i b_i c_k \overset{A1)}{=} a_i c_k^{-1} b_i c_k$

□

Using the diagram given in Montesinos' paper [9], we can write down in terms of our monoid, the formula for every twist-spun knots as follows.

Proposition 3.10.
Let a classical knot $\widehat{K}$ be the plat closure of the braid K on $2m+1$ strands, then

$$\tau^n(\widehat{K}) = \left[\left(\prod_{i=1}^{m} a_{2i}\right) K \left(\prod_{i=1}^{m} b_{2i}\right) K^{-1}\Delta_{2m+1}^{2n}\right].$$

Example 3.11. Let us now see how to unknot n-twist-spun trivial knot on 3 strands. We have

$\tau^n(\widehat{c_1^{-1}}) = [a_2(c_1)^{-1}b_2c_1\Delta^{2n}]_3 \overset{\text{A14)}}{=} [a_2(c_1)^{-1}b_2c_1] \overset{\text{C1)}}{=}$
$= [c_1a_2(c_1)^{-1}b_2] \overset{\text{A3)}}{=} [c_2^{-1}a_1c_2b_2] \overset{\text{C1)}}{=} [c_2b_2c_2^{-1}a_1]_3 \overset{\text{C2)}}{=} [c_2b_2c_2^{-1}]_2 \overset{\text{A2), A1)}}{=}$
$= [b_2]_2 \overset{\text{C2)}}{=} [1]_1$.

We now use this algebraic method to relate presented theorems by Zeeman 2.3 and Litherland 2.4.

Theorem 3.12. *Equations $\tau^{-m}(K) = \tau^m(K)$ and $(\tau^m(K))^* = \tau^m(K)$ are equivalent. More generally we have:*

$$\tau^n(K) = \left(\tau^{-n}(K)\right)^*.$$

Proof. Let $K = \widehat{R}$, then

$\tau^n(K) = \left[\left(\prod_{i=1}^m a_{2i}\right) R \left(\prod_{i=1}^m b_{2i}\right) R^{-1}\Delta_{2m+1}^{2n}\right] \overset{\text{mirror image}}{=}$
$= \left[\Delta_{2m+1}^{-2n} R \left(\prod_{i=1}^m b_{2i}\right) R^{-1} \left(\prod_{i=1}^m a_{2i}\right)\right]^* \overset{\text{relation C1)}}{=}$
$= \left[\left(\prod_{i=1}^m a_{2i}\right) \Delta_{2m+1}^{-2n} R \left(\prod_{i=1}^m b_{2i}\right) R^{-1}\right]^* \overset{\text{relation A13)}}{=}$
$= \left[\left(\prod_{i=1}^m a_{2i}\right) R \left(\prod_{i=1}^m b_{2i}\right) R^{-1}\Delta_{2m+1}^{-2n}\right]^* = \left(\tau^{-n}(K)\right)^*$ □

3.1. Index of a surface singular braid.

Definition 3.13. The *singular braid index* of knotted surface F, denoted by $Ind_S(F)$, is the minimum degree among all surface singular braids, that its closure gives marked diagram of surface equivalent to F.

It is easily seen that if $Ind_S(F) = 1$ (i.e. $F = [1]_1$) then F is standard unknotted 2-sphere $\mathbb{S}^2$. We will investigate this notion further.

Theorem 3.14. *If $Ind_S(F) = 2$ then there are exactly six types of knotted surfaces F. Moreover, there exist infinitely many knotted surface types F such that $Ind_S(F) = 3$.*

Proof. Let us consider elements of CSB_2, from the relation A2) it follows that all of them are commutative. So each surface is in the form $[a_1^\alpha b_1^\beta c_1^\gamma c_1^{-\delta}]_2$, for $\alpha, \beta, \gamma, \delta \geq 0$. By relations A09) and A10) it follows that all this knotted surfaces are in the form $[a_1^\alpha b_1^\beta c_1^\gamma c_1^{-\delta}]_2$, where $\alpha, \beta \in \{0,1\}$ and $\gamma, \delta \geq 0$.

If $\alpha \cdot \beta = 1$ then by relation A11) we have $\gamma, \delta \in \{0,1\}$ giving us two types of surfaces: $[a_1b_1]$ and $[a_1b_1c_1]$. If $\alpha \cdot \beta = 0$ then one of resolutions L_+ or L_- is a diagram of torus link $T(2, \delta-\gamma)$. This classical link must be trivial by Theorem 2.7, so we have that $|\delta - \gamma| \in \{0,1\}$ and using the relation A1) if needed, we have that $\gamma, \delta \in \{0,1\}$. This gives us four more types of surfaces: $[1]_2, [c_1], [a_1c_1], [a_1c_1^{-1}]$.

All of those above mention six types of surface links are known to be mutually distinct.

As for the case $Ind_S(F) = 3$, we can take a family of surface knots $\tau^2(T(2,k))$ for odd prime k. They are mutually distinct (see paper [1]) and they are 3-strand closure of surface singular braid word $a_2 c_1^{-k} b_2 c_1^k \Delta^4$.

□

3.2. **Minimal number of Reidemeister III moves.**

We now give a family of pairs of diagrams of classical links such that deforming one of them to the other requires minimum four Reidemeister III moves.

Theorem 3.15. *There exists a family of classical diagrams $D_{n,k}$ for $n \geq 2$ and odd $k \geq 3$ of 2-component links with $6n + 2k$ crossings such that there is at least four Reidemeister III moves required to transform the diagram $D_{n,k}$ into trivial diagram without any crossings.*

Proof. Let us consider a diagram $D_{n,k}$ as a (modified) plat closure of word $c_1^k \Delta^{2n} c_1^{-k}$, presented in the Figure 5 .

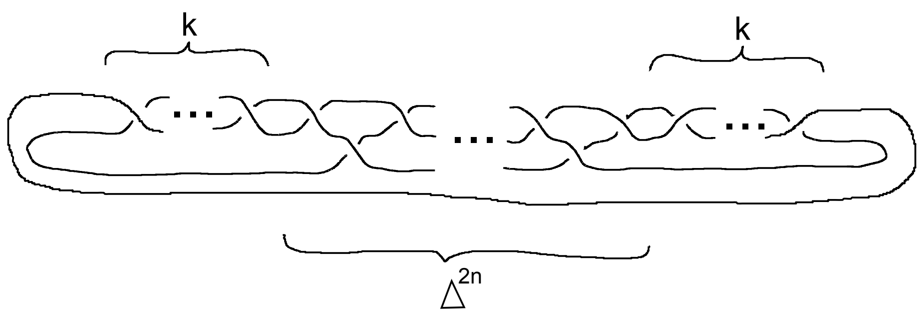

FIGURE 5. diagram $D_{n,k}$

From the Proposition 2.1 we know that every Reidemeister III move in a motion picture stills corresponds to one triple point in a surface diagram. By uniqueness of knotted surface type from a given marked diagram (Theorem 2.6) it follows that, it is sufficient to prove that there exists a knotted surface F, such that $L_-(F)$ is the $D_{n,k}$ diagram and $L_+(F)$ can be unlinked without any Reidemeister III move; finally that the surface F has at least four triple points in every in every projection to $\mathbb{R}^3$.

The latter one follows from combining theorems of Satoh from paper [10] and Cochran from paper [6], for F being the n-twist-spun torus knot $T(2,k)$ for $n \geq 2$ and odd integer $k \geq 3$.

To prove the ability of unlink $L_+(F)$ with only using Reidemeister I-st or II-nd moves, we will proceed directly. Diagram $L_+(F)$ is at the beginning a (modified) plat closure of the braid word $c_1^{-k} c_1^k \Delta^{2n}$, after obvious reduction of the word $c_1^{-k} c_1^k$,

we sequentially reduce every plat closure (from one side) of expression Δ^2 as in the Figure 6. We receive at the end a diagram of two disjoint circles on the plane.

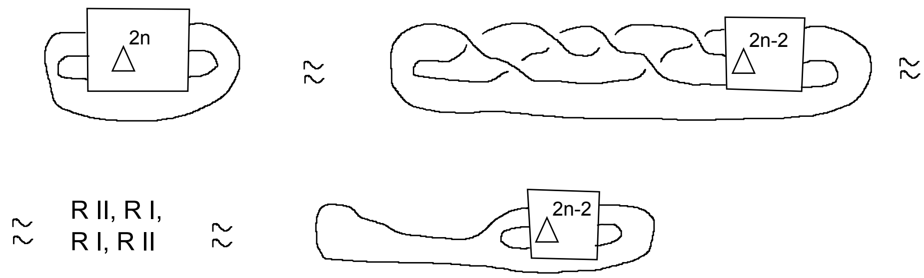

FIGURE 6. part of reduction of $L_+(F)$

Finally we see that $L_-(F)$ is a (modified by planar isotopy) presented in the Figure 5 diagram $D_{n,k}$, because we have that $\tau^n(T(2,k)) = [a_2c_1^{-k}b_2c_1^k\Delta^{2n}]$. □

Definition of twist-spun torus knots raises an interesting question about duality of its parameters. For example $\tau^n(T(2,k))$ and $\tau^k(T(2,n))$ for $n = 1$ or $k = 1$ gives the same (unknotted) surface knot. But in the case k, n are distinct odd integers greater than one, quandle cocycle invariants do not distinguish them (see paper [1]). In our algebraic language we state the following.

Question 3.16. For what odd different integers $k, n > 1$ we have $[a_2c_1^{-k}b_2c_1^k\Delta^{2n}] = [a_2c_1^{-n}b_2c_1^n\Delta^{2k}]$?

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REFERENCES

[1] S. Asami and S. Satoh, *An infinite family of non-invertible surfaces in 4-space*, Bull. London Math. Soc. **37** (2005) 285-296.

[2] J. C. Baez, *Link invariants of finite type and perturbation theory*, Lett. Math. Phys. **26** (1992) 43-51.

[3] J. Birman, *New points of view in knot theory*, Bull. Am. Math. Soc. **28** (1993) 253–287.

[4] J. S. Carter, M. Elhamdadi, M. Saito and S. Satoh, *A lower bound for the number of Reidemeister moves of type III*, Topology Appl. **153** (2006) 2788-2794.

[5] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Math. Surveys and Monographs **55**, American Mathematical Society (1998).

[6] T. Cochran, *Ribbon knots in* $\mathbb{S}^4$, J. London Math. Soc. **28** (1983) 563-576.

[7] R. A. Litherland, *Symmetries of twist-spun knots*, in "Knot theory and manifolds" (Vancouver, B.C., 1983), 97–107, Lecture Notes in Math., **1144**, Springer, Berlin-New York, 1985.

[8] S. J. Jr. Lomonaco, *The Homotopy Groups of Knots I. How to Compute the Algebraic 2-Type*, Pacific J. Math. **95** (1981) 349–390.

[9] J. M. Montesinos, *A note on twist spun knots*, Proc. Amer. Math. Soc. **98** (1986) 180-184.

[10] S. Satoh, *No 2-knot has triple point number two or three*, Osaka J. Math. **42** (2005) 543-556.

[11] F. J. Swenton, *On a calculus for 2-knots and surfaces in 4-space*, J. Knot Theory Ramifications **10** (2001) 1133-1141.

[12] K. Yoshikawa, *An enumeration of surfaces in four-space*, Osaka J. Math. **31** (1994) 497-522.
[13] E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965) 417-495.

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